Int. J. Solids Structures Vol. 23, No. 9, pp. 1219–1224, 1987
Printed in Great Britain.

0020–7683/87 $3.00+ .00

THE SHRINK FIT WITH ELASTIC–PLASTIC HUB EXHIBITING CONSTANT YIELD STRESS FOLLOWED BY HARDENING

U. GAMER
Institut für Mechanik, Technische Universität Wien, Karlsplatz 13, A-1040 Wien, Austria

(*Received* 1 *August* 1986; *in revised form* 4 *November* 1986)

Abstract—The subject of the investigation is the distribution of stress, displacement and plastic strain in a rotating shrink fit, the hub material of which undergoes plastic deformation with constant stress up to a certain plastic strain and linear hardening for larger strain.

INTRODUCTION

Shrink fits are applied widely in mechanical engineering since they transmit high moments at low production costs. To better utilize the hub material, plastic deformation of the hub is admitted frequently. The investigation of the elastic–plastic stress distribution was started in 1944 by Lundberg[1]. His calculation is based on von Mises' yield criterion in connection with Hecky's deformation theory. Unfortunately, the application of Lundberg's results is rather complicated. An exact solution of the elastic–plastic shrink fit problem based on Tresca's yield criterion and the associated flow rule was derived by Kollmann for shrink fits at rest[2] and for rotating shrink fits[3]. As done by most investigators, he used a circular disk included in an annulus or two annuli as a model for the shrink fit. Later on, Kollmann's results were generalized to elastic–plastic materials with linear hardening[4, 5]. Further progress was made recently, when it was shown that hub material with an arbitrary non-linear hardening law can be taken into account with very little numerical calculation[6, 7]. In this paper, the new results are applied to a hub material with perfectly plastic behaviour up to a certain plastic strain and linear hardening for larger strain. The inclusion is a circular disk (Fig. 1). Of course, this problem can also be solved by modifying the derivation in use hitherto.

BASIC EQUATIONS

In the following, the basic equations of the problem are compiled[6, 7]. Provided that, as a consequence of the inequalities $\sigma_r \leqslant 0$ and $\sigma_\theta \geqslant 0$, the flow rule in the entire plastic region of the hub read $d\varepsilon_r^p/d\varepsilon_\theta^p = -1$ and $d\varepsilon_z^p = 0$, the displacement in inclusion and hub is given by

$$Eu = (1-\nu)(\sigma_r + \tfrac{1}{4}\rho\omega^2 r^2)r + \frac{C}{r} \tag{1}$$

with

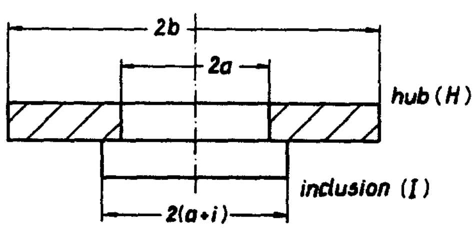

Fig. 1. Shrink fit geometry prior to assemblage.

$$C^{\mathrm{I}} = 0, \qquad C^{\mathrm{H}} = Eai \tag{2}$$

for the inclusion (I) and hub (H), respectively. In these relations, i is the interference and ω the angular speed.

The stresses in the inclusion and in the elastic part of the hub near the edge have the forms[8]

$$\sigma_r = -\frac{A}{r^2} + B - \tfrac{1}{8}(3+\nu)\rho\omega^2 r^2 \tag{3}$$

$$\sigma_\theta = \frac{A}{r^2} + B - \tfrac{1}{8}(1+3\nu)\rho\omega^2 r^2. \tag{4}$$

The plastic strains in the entire plastic region of the hub adjacent to the interface read

$$E\varepsilon_\theta^p = -E\varepsilon_r^p = -\sigma_Y + \tfrac{1}{4}(1-\nu)\rho\omega^2 r^2 + \frac{Eai}{r^2} \tag{5}$$

where the yield stress, σ_Y, grows in some manner with the equivalent plastic strain, ε_E^p. In the present case, consideration of the increment of plastic work and the yield condition

$$\sigma_\theta - \sigma_r = \sigma_Y \tag{6}$$

leads to $\varepsilon_E^p = \varepsilon_\theta^p$. The strains are presumed infinitesimal; the plastic strain is the difference of the total strain and its elastic part.

Inserting the yield condition in the equation of motion, one obtains after integration the stresses

$$\sigma_r = \int \frac{\sigma_Y}{r}\,\mathrm{d}r - \tfrac{1}{2}\rho\omega^2 r^2 + D \tag{7}$$

$$\sigma_\theta = \sigma_r + \sigma_Y. \tag{8}$$

Independently of the hardening law, at the elastic–plastic border, $r = z$, the yield stress equals the initial yield limit, σ_0. Therewith, eqn (5) yields

$$Z^2 = 2\frac{1-\sqrt{(1-(1-\nu)\bar{\Omega}^2\bar{I})}}{(1-\nu)\bar{\Omega}^2} \tag{9}$$

where $\bar{I} := Eai/(\sigma_0 b^2)$ denotes non-dimensional interference, $\bar{\Omega}^2 := \rho\omega^2 b^2/\sigma_0$ non-dimensional angular speed and $Z := z/b$ the non-dimensional elastic–plastic border radius with a and b as inner and outer radius of the hub, respectively. For shrink fits at rest, eqn (9) simplifies to

$$Z^2 = \bar{I}. \tag{10}$$

THE PLASTIC REGION

In the problem under consideration, perfectly plastic behaviour is assumed up to a characteristic plastic strain, $\varepsilon_E^p = \varepsilon_0$, and linear hardening for larger strain

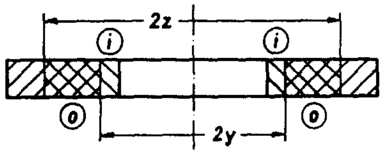

Fig. 2. Partially plasticized hub with inner part (i) and outer part (o) of the plastic region.

$$\sigma_Y = \begin{cases} \sigma_0, & \varepsilon_E^p \leqslant \varepsilon_0 \\ \sigma_0[1+\eta(\varepsilon_E^p-\varepsilon_0)], & \varepsilon_E^p \geqslant \varepsilon_0 \end{cases} \tag{11}$$

where η is the hardening parameter. With $\sigma_Y = \sigma_0$ and $\varepsilon_\theta^p = \varepsilon_0$, eqn (5) delivers the internal border radius, $r = y$, between the two parts of the plastic region

$$Y^2 = 2\frac{\bar{\varepsilon}_0+1-\sqrt{((\bar{\varepsilon}_0+1)^2-(1-\nu)\bar{\Omega}^2\bar{I})}}{(1-\nu)\bar{\Omega}^2} \tag{12}$$

with $\bar{\varepsilon}_0 := E\varepsilon_0/\sigma_0$ and $Y := y/b$ (Fig. 2). For shrink fits at rest, eqn (12) is reduced to

$$Y^2 = \frac{\bar{I}}{\bar{\varepsilon}_0+1}. \tag{13}$$

For $y \leqslant r \leqslant z$, one obtains from eqns (7) and (8) together with eqn (11) the well-known stresses

$$\sigma_r = \sigma_0 \log\frac{r}{b} - \tfrac{1}{2}\rho\omega^2 r^2 + D^{\circ} \tag{14}$$

$$\sigma_\theta = \sigma_r + \sigma_0 \tag{15}$$

in the outer part (o) of the plastic region.

Inserting the equivalent plastic strain, ε_E^p, which equals the circumferential plastic strain, eqn (5), into the hardening law, eqn (11), and solving for the yield stress, σ_Y, one obtains

$$\bar{\sigma}_Y = \frac{1}{1+H}\left[1-H\bar{\varepsilon}_0+\tfrac{1}{4}(1-\nu)H\bar{\Omega}^2X^2+\frac{H\bar{I}}{X^2}\right] \tag{16}$$

with the normalized hardening parameter $H := \sigma_0\eta/E$, the non-dimensional yield stress $\bar{\sigma}_Y := \sigma_Y/\sigma_0$ and the non-dimensional radius $X = r/b$.

Now, eqns (7) and (8) yield the stresses

$$\bar{\sigma}_r = \frac{1}{1+H}\left\{(1-H\bar{\varepsilon}_0)\log X-\tfrac{1}{8}[4+(3+\nu)H]\bar{\Omega}^2X^2-\frac{1}{2}\frac{H\bar{I}}{X^2}\right\}+\bar{D}^{i} \tag{17}$$

$$\bar{\sigma}_\theta = \bar{\sigma}_r + \bar{\sigma}_Y \tag{18}$$

in the inner part (i), $a \leqslant r \leqslant y$, of the plastic region.

CONDITIONS AND RESULTS

The starting point is the elastic region of the hub near the edge. Forming the circumferential strain, ε_θ, with the help of Hooke's law and expressions (3) and (4) on the one hand and using eqn (1) on the other hand, one finds by comparison $A = C/2$, i.e.

$$A^{\mathrm{I}} = 0, \qquad A^{\mathrm{H}} = \tfrac{1}{2}Eai. \tag{19}$$

Therewith, the condition of vanishing radial stress at the free edge of the hub yields

$$B^{\mathrm{H}} = \tfrac{1}{8}(3+\nu)\rho\omega^2 b^2 + \tfrac{1}{2}E\frac{ai}{b^2}. \tag{20}$$

A^{H} and B^{H} agree with the constants contained in the expressions for stress and displacement of the hub for unrestricted elastic behaviour. It is well known that the occurrence of plastic flow at the interface does not influence the distribution of stress and displacement in the elastic part of the hub[4, 5].

The solution of the problem is completed by adapting, with the help of the condition of continuity of radial stress, the outer part of the plastic region to the elastic region, the inner part of the plastic region to the outer part and, finally, the inclusion to the inner part of the plastic region of the hub. The system of equations is completely uncoupled; each equation contains a single unknown.

In non-dimensional form, as used in the stresses normalized by σ_0, the results are

$$\bar{D}^{\mathrm{o}} := \frac{D^0}{\sigma_0} = -\log Z + \tfrac{1}{8}\Omega^2[3+\nu+(1-\nu)Z^2] - \tfrac{1}{2}\bar{I}\left(\frac{1}{Z^2} - 1\right) \tag{21}$$

$$\bar{D}^{\mathrm{i}} = \log\frac{Y}{Z} - \frac{1-H\bar{\varepsilon}_0}{1+H}\log Y + \tfrac{1}{8}\Omega^2\left[3+\nu+(1-\nu)Z^2 - \frac{(1-\nu)H}{1+H}Y^2\right] + \tfrac{1}{2}\bar{I}\left[\frac{H}{1+H}\frac{1}{Y^2} - \frac{1}{Z^2} + 1\right] \tag{22}$$

$$\bar{B}^{\mathrm{I}} := \frac{B^{\mathrm{I}}}{\sigma_0} = \log\frac{Y}{Z} + \frac{1-H\bar{\varepsilon}_0}{1+H}\log\frac{Q}{Y} + \tfrac{1}{8}\Omega^2\left[3+\nu+(1-\nu)Z^2 - \frac{1-\nu}{1+H}(HY^2+Q^2)\right] + \tfrac{1}{2}\bar{I}\left[\frac{H}{1+H}\left(\frac{1}{Y^2} - \frac{1}{Q^2}\right) - \frac{1}{Z^2} + 1\right] \tag{23}$$

with the radii ratio $Q := a/b$.

This solution applies, provided that $Z \leqslant 1$, or

$$\Omega^2 \leqslant \frac{4}{1-\nu}(1-\bar{I}) \tag{24}$$

and that the radial stress does not become positive inside the plastic region, or

$$\Omega^2 \leqslant \frac{4}{3+\nu}\left(1 - \frac{1-\nu}{3+\nu}\bar{I}\right) \tag{25}$$

and that $Y \geqslant Q$. The shrink fit rotating with supercritical angular speed surpassing the one given by expression (25) was treated for perfectly plastic hub material in Ref. [9]. The case of the fully plasticized hub will be dealt with in the next section.

THE SHRINK FIT WITH FULLY PLASTICIZED HUB

The elastic region of the hub is dropped. The condition of vanishing radial stress at the free edge yields

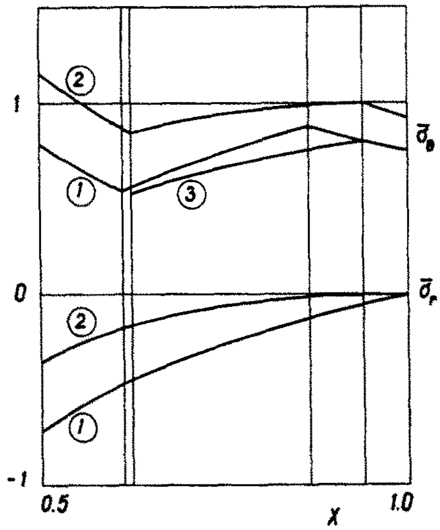

Fig. 3. Radial and circumferential stress in the hub after assemblage ①, rotating with critical angular speed ② and circumferential stress after rotation ③.

$$\bar{D}^{o} = \tfrac{1}{2}\bar{\Omega}^{2}. \tag{26}$$

From the condition of continuity of radial stress, one obtains successively

$$\bar{D}^{i} = \log Y - \frac{1-H\bar{\varepsilon}_0}{1+H}\log Y + \tfrac{1}{8}\bar{\Omega}^2\left[4 - \frac{(1-\nu)H}{1+H}Y^2\right] + \tfrac{1}{2}\bar{I}\frac{H}{1+H}\frac{1}{Y^2} \tag{27}$$

and

$$\bar{B}^{i} = \log Y + \frac{1-H\bar{\varepsilon}_0}{1+H}\log\frac{Q}{Y} + \tfrac{1}{8}\bar{\Omega}^2\left[4 - \frac{1-\nu}{1+H}(HY^2+Q^2)\right] + \tfrac{1}{2}\bar{I}\frac{H}{1+H}\left(\frac{1}{Y^2}-\frac{1}{Q^2}\right). \tag{28}$$

The slope of σ_r at the edge, $r = b$, may not become negative. This is ensured by the condition

$$\bar{\Omega} \leqslant 1. \tag{29}$$

NUMERICAL RESULTS

Figure 3 shows, for $\nu = 1/3$, $H = 1$, $\bar{\varepsilon}_0 = 1$, $Q = 1/2$ and $\bar{I} = 3/4$ the stresses $\bar{\sigma}_r$ and $\bar{\sigma}_\theta$ in the hub due to assemblage, marked by ①, the stresses in the hub rotating with the critical angular speed $\bar{\Omega}^2 = 51/50$ according to expression (25), marked by ②, and the circumferential stress in the hub at rest after rotation, marked by ③. In the elastic part of the hub, the stresses before and after rotation coincide since, as mentioned already, there is no influence of plastic flow on the stress distribution in this region. The radial stress in the final stage lies slightly above the one caused by assemblage; at the interface the difference amounts to 0.042. Deceleration is not connected with additional plastic flow[10].

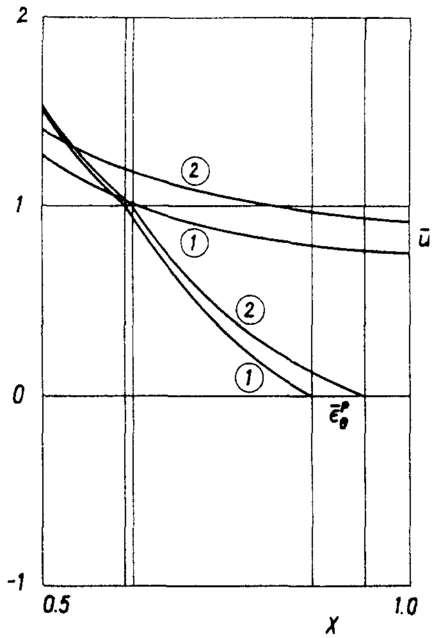

Fig. 4. Displacement and circumferential plastic strain in the hub after assemblage ① and rotating with critical angular speed ②.

On Fig. 4, for the same parameters, displacement, $\bar{u} := Eu/(\sigma_0 b)$, and circumferential plastic strain, $\bar{\varepsilon}_\theta^p := E\varepsilon_\theta^p/\sigma_0$, in the hub is plotted. Again, ① belongs to assemblage and ② to rotation with critical speed. During deceleration, the plastic strain is not affected any more. Due to plastic flow during acceleration, the displacement in the previously plastic region of the hub at rest after rotation is slightly higher than the displacement caused by assemblage. At the interface, the difference is 0.014. In the elastic region, the displacement after assemblage and after rotation agrees.

REFERENCES

1. G. Lundberg, Die Festigkeit von Preßsitzen. *Kugellager* **19**, 1–11 (1944).
2. F. G. Kollmann, Die Auslegung elastisch-plastisch beanspruchter Querpreßverbände. *Forsch. Ing.-Wes.* **44**, 1–11 (1978).
3. F. G. Kollmann, Rotating elasto-plastic interference fits. *Trans. Am. Soc. Mech. Engrs, J. Mech. Des.* **103**, 61–66 (1981).
4. U. Gamer and R. H. Lance, Elastisch–plastische Spannungen im Schrumpfsitz. *Forsch. Ing.-Wes.* **48**, 192–198 (1982).
5. U. Gamer, The rotating elastic–plastic shrink fit with hardening. *Acta Mech.* **61**, 15–27 (1986).
6. U. Gamer, Die teilplastizierte Nabe eines Preßverbandes. *Z. Angew. Math. Mech.* **67**, 65–66 (1987).
7. U. Gamer, The shrink fit with nonlinearly hardening hub. *Trans. Am. Soc. Mech. Engrs, J. Appl. Mech* (1987), in press.
8. S. P. Timoshenko and J. N. Goodier, *Theory of Elasticity*. McGraw-Hill, New York (1970).
9. U. Gamer and F. G. Kollman, A theory of rotating elasto-plastic shrink fits. *Ing.-Arch.* **56**, 254–264 (1986).
10. U. Gamer, Die Spannungen im elastisch–plastischen Preßverband nach Rotation. *Forsch. Ing.-Wes.* (1987), in press.